固有値が1の二階非線形差分方程式に帰着される、ある関数方程式について(現象からの関数方程式)

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固有値が１の二階非線形差分方程式に帰着される、ある関数方程式について

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1 Introduction

We consider the following functional equation

$$\Psi(X(x,\Psi(x)))=Y(x,\Psi(x)), \tag{1.1}$$

where $X(x,y)$, $Y(x,y)$ are function of $(x,y)\in\mathbb{C}^2$, holomorphic in a neighborhood U of $(0,0)$.

Here we suppose that $X(x,y)$ and $Y(x,y)$ are written in a neighborhood U of $(0,0)$ as :

$$\begin{cases} X(x,y)=x+y+\sum_{i+j\geq 2}c_{ij}x^iy^j=x+X_1(x,y),\\ Y(x,y)=y+\sum_{i+j\geq 2}d_{ij}x^iy^j=y+Y_1(x,y). \end{cases} \tag{1.2}$$

For the equation (1.1), in which X and Y are written as follows

$$\begin{cases} X(x,y)=\lambda x+\lambda' y+\sum_{i+j\geq 2}c_{ij}x^iy^j=\lambda x+X_1(x,y),\\ Y(x,y)=\mu y+\sum_{i+j\geq 2}d_{ij}x^iy^j=\mu y+Y_1(x,y), \end{cases}$$

we considered the case $|\lambda|>1,\lambda'=0$ and $|\lambda|<1,\lambda'=0$ in [5], the case $\lambda=\mu$, $|\lambda|\neq 1$, $\lambda'=0$ and $\lambda=\mu$, $|\lambda|\neq 1$, $\lambda'=1$ in [8], $\lambda=\mu=1,\lambda'=0$ in [6], the case $\lambda=1,|\mu|=1$, $\lambda'=0$ in [7]. In this present paper, we consider the equation (1.1) in the case $\lambda=\mu=\lambda'=1$.

When we consider a nonlinear simultaneous system of difference equations:

$$\begin{cases} x(t+1)=X(x(t),y(t)),\\ y(t+1)=Y(x(t),y(t)), \end{cases} \tag{1.3}$$

we can reduce it to the following single equation (see [8])

$$x(t+1) = X\Big(x(t), \Psi(x(t))\Big),$$

making use of the equation (1.1). In [3], Kimura consider the first order nonlinear difference equation, in which eigenvalue is equal to 1. If we can have a solution of (1.1), then we have an analytic solution of (1.3) making use of the theorem in [3].

In this present paper we have the following theorem 1.

Theorem 1. *Suppose $X(x,y)$ and $Y(x,y)$ are defined in (1.2). Suppose $d_{20}=0$,*

$$\frac{2c_{20}+d_{11}\pm\sqrt{(2c_{20}-d_{11})^2+8d_{30}}}{4}\in\mathbb{R},\ \frac{2c_{20}+d_{11}+\sqrt{(2c_{20}-d_{11})^2+8d_{30}}}{4}<0, \tag{1.4}$$

and we assume the following conditions,

$$(g_0^+(c_{20},d_{11},d_{30})+c_{20})n \neq c_{20}-d_{11}-g_0^+(c_{20},d_{11},d_{30}) \tag{1.5}$$

$$(g_0^-(c_{20},d_{11},d_{30})+c_{20})n \neq c_{20}-d_{11}-g_0^-(c_{20},d_{11},d_{30}) \tag{1.6}$$

for all $n\in\mathbb{N}$, $(n\geqq 4)$, where

$$g_0^{\pm}(c_{20},d_{11},d_{30}) = \frac{-(2c_{20}-d_{11})\pm\sqrt{(2c_{20}-d_{11})^2+8d_{30}}}{4}, \tag{1.7}$$

respectively, then we have a formal solution $\Psi(x)=\sum_{n\geqq 2}^{\infty}a_nx^n$ of (1.1). Further, for any κ, $0<\kappa\leqq\frac{\pi}{2}$, there are a $\delta>0$ and a solution $\Psi(x)$ of (1.1), which is holomorphic and can be expanded asymptotically as

$$\Psi(x)\sim\sum_{n=2}^{\infty}a_nx^n, \tag{1.8}$$

in the following domain $D(\kappa,\delta)$,

$$D(\kappa,\delta)=\{x\,;\,|\arg x|<\kappa,\ 0<|x|<\delta\}. \tag{1.9}$$

2 Proof of the theorem

2.1 Determination of a formal solution

At first, we put a formal solution of (1.1) as $\Psi(x)=\sum_{n=1}^{\infty}a_nx^n$. To determine coefficients a_m, we substitute $\Psi(x)=\sum_{n=1}^{\infty}a_nx^n$ into (1.1) with (1.2), and have

$$\sum_{n=1}^{\infty}a_n\Big\{(1+a_1)x+\sum_{m=2}^{\infty}a_mx^m+\sum_{i+j\geqq 2}c_{ij}\Big(\sum_{k_1,\cdots,k_j\geqq 1}a_{k_1}\cdots a_{k_j}x^{k_1+\cdots+k_j+i}\Big)\Big\}^n$$
$$=\sum_{n=1}a_nx^n+\sum_{i'+j'\geqq 2}d_{i'j'}\Big(\sum_{k_1,\cdots,k_j\geqq 1}a_{k_1}a_{k_2}\cdots a_{k_j}\cdot x^{k_1+\cdots+k_j+i}\Big). \tag{2.1}$$

We compare the coefficients of x^n, $(n = 1, 2, \cdots)$ in (2.1), then we have

$$\begin{cases} x^1 : a_1 = 0, \\ x^2 : d_{20} = 0, \\ x^3 : a_2\{2a_2 + (2c_{20} - d_{11})\} = d_{30}, \\ x^4 : a_3\{5a_2 + (3c_{20} - d_{11})\} = -2a_2(c_{30} + c_{11}a_2) - a_2(a_2 + c_{20})^2 + d_{21}a_2 + d_{02}a_2^2 + d_{40}, \\ \quad \cdots \quad , \\ x^n : a_{n-1}\{(n+1)a_2 + (n-1)c_{20} - d_{11}\} = f_{n-1}(a_2, a_3, \cdots, a_{n-2}, c_{ij}, d_{i'j'}), \ (n \geqq 4). \end{cases}$$

Where $f_n(a_2, a_3, \cdots, a_{n-2}, c_{ij}, d_{i'j'})$ are polynomials for $a_2, a_3, \cdots, a_{n-2}, c_{ij}, d_{i'j'}$, $i+j \leqq n-1$, $i'+j' \leqq n-1$.

From the coefficients of x and x^2, we have $a_1 = 0$ and $d_{20} = 0$. From the coefficients of x^3 we have

$$a_2 = g_0^+(c_{20}, d_{11}, d_{30}),\ g_0^-(c_{20}, d_{11}, d_{30}).$$

From the coefficients of x^n $(n \geqq 4)$, we have

$$a_{n-1}^+\{(g_0^+(c_{20}, d_{11}, d_{30}) + c_{20})n - c_{20} + d_{11} + g_0^+(c_{20}, d_{11}, d_{30})\} = f_{n-1}(a_2, a_3, \cdots, a_{n-2}, c_{ij}, d_{i'j'}),$$

or

$$a_{n-1}^-\{(g_0^-(c_{20}, d_{11}, d_{30}) + c_{20})n - c_{20} + d_{11} + g_0^-(c_{20}, d_{11}, d_{30})\} = f_{n-1}(a_2, a_3, \cdots, a_{n-2}, c_{ij}, d_{i'j'}).$$

From the following assumption (1.5) and (1.6), for all $n \in \mathbb{N}$, $(n \geqq 4)$, we have

$$a_{n-1}^\pm = \frac{f_{n-1}(a_2, a_3, \cdots, a_{n-2}, c_{ij}, d_{i'j'})}{(n+1)g_0^\pm(c_{20}, d_{11}, d_{30}) + (n-1)c_{20} - d_{11}}, \ (n \geqq 4), \tag{2.2}$$

respectively. Therefore we can decide a formal solution

$$\Psi(x) = \sum_{n=2}^{\infty} a_n x^n. \tag{2.3}$$

2.2 Existence of a solution $\Psi(x)$

In this subsection we prove the existence a solution $\Psi(x)$ of (1.1) under the condition (1.4), (1.5) and (1.6).

2.2.1 Map T

Put

$$u - Y(x, y) = 0, \tag{2.4}$$

$$f(u, x, y) = u - \{y + \sum_{i'+j' \geqq 2} d_{i'j'} x^{i'} y^{j'}\}. \tag{2.5}$$

Since $f(0,0,0)=0$, $\left.\frac{\partial f}{\partial y}\right|_{x=y=u=0} = -1 \neq 0$, thus, we obtain an inverse function $H(x,u)$, such that

$$y = H(x,u) = u + H_1(x,u),\ H_1(x,u) = \sum_{i+j \geqq 2} r_{ij} x^i y^j,$$

defined in $|x| < \epsilon_1$, $|u| < \epsilon_2$, where ϵ_1 and ϵ_2 are small positive constants. The range of $H(x,u)$ contains a disc $|y| < \epsilon_3$. Let $\epsilon = \min(\epsilon_1, \epsilon_2, \epsilon_3)$. Then the equation (1.1) is equivalent to the following equation (2.6)

$$\Psi(x) = H\Big(x, \Psi\big(X(x, \Psi(x))\big)\Big), \text{ for } |x| < \epsilon. \tag{2.6}$$

Let κ be a number such that $0 < \kappa < \pi/2$. Take a positive integer $N > 3$. Let $g_N(x) = \sum_{n=2}^{N} a_n x^n$ be the truncation of the formal solutions of (2.3). Put

$$\mathfrak{F} = \mathfrak{F}(N, K, \delta) = \{\phi(x)\,;\, \phi(x) \text{is holomorphic and satisfies}$$
$$|\phi(x)| \leqq K|x|^N \text{ and } |g_N(x)| + K|x|^N < \delta, \text{in } D(\kappa,\delta)\}$$

where N, K and δ are positive constants to be determined later. Note that K and δ may be depend on N, and will be expressed, sometimes, as $K(N)$, $\delta(N)$, respectively.

Put $v = X(x, g_N(x) + \phi(x))$, we have

$$|v| = |x| \cdot |1 + (a_2 + c_{20})\mathbb{R}[x] + \text{higher terms}|, \tag{2.7}$$
$$\arg[v] = \arg[x] + \arg[1 + (a_2 + c_{20})x + x^2 F_0(x, \phi(x))]. \tag{2.8}$$

From the condition (1.4), we have $a_2 + c_{20} \leqq \frac{2c_{20}+d_{11}+\sqrt{(2c_{20}-d_{11})^2+8d_{30}}}{4} < 0$. Since, $-\pi/2 < \arg[x] < \pi/2$, further if δ is sufficiently small, then we have $|x|/2 < |v| < |x|$ and $|\arg[v]| < |\arg[x]|$, (see Figure 1).

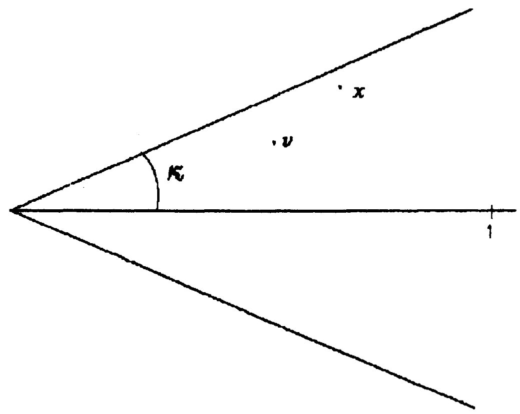

Figure 1

Thus, if $x \in D(\kappa,\delta)$, then $v \in D(\kappa,\delta)$ and $\phi\Big(X(x, g_N(x) + \phi(x))\Big)$ is defined for $\phi(x) \in \mathfrak{F}$. Hence we can define the following map T, for a $\phi(x) \in \mathfrak{F}$,

$$T[\phi](x) = H\Big(x, g_N\big(X(x, g_N(x) + \phi(x))\big) + \phi\big(X(x, g_N(x) + \phi(x))\big)\Big) - g_N(x). \tag{2.9}$$

If there is a unique fixed point $\phi_0(x)$ in $\mathfrak{F}$ and further it is independent of N, then we have a solution $\Psi(x)$ of (1.1) which is holomorphic and can be expanded asymptotically as in (1.8) in the domain $D(\kappa,\delta)$, such that $\Psi(x) = g_N(x) + \phi_0(x)$.

2.2.2 Existence of a fixed point of T

From (2.9) we have

$$\begin{aligned} T[\phi](x) &= \{H\Big(x,(g_N+\phi)(X(x,g_N(x)+\phi(x)))\Big) - H\Big(x,g_N(X(x,g_N(x)+\phi(x)))\Big)\} \\ &\quad + \{H\Big(x,g_N(X(x,g_N(x)+\phi(x)))\Big) - H\Big(x,g_N(X(x,g_N(x)))\Big)\} \\ &\quad + \{H\Big(x,g_N(X(x,g_N(x)))\Big) - g_N(x)\} \\ &= U[\phi](x) + V[\phi](x) + W[\phi](x). \end{aligned} \tag{2.10}$$

Since $g_N(x)$ is the truncated formal solution, we have

$$|W(x)| \leqq K_1(N)|x|^{N+1}, \tag{2.11}$$

for a constant $K_1(N)$ which is dependent on N. Put $u_1 = g_N\big(X(x,g_N(x)+\phi(x))\big)$, $u_2 = g_N\big(X(x,g_N(x))\big)$. Then we have

$$\begin{aligned} |u_1-u_2| &\leqq 2|a_2|(1+|a_2|)|x|\{|\phi(x)|(1+K_2(N)|x|)\}, \\ |1+r_{11}x+r_{02}(u_1+u_2) &+ \text{higher terms}| \leqq 2 \end{aligned}$$

Therefore, we have

$$|V[\phi](x)| = |H(x,u_1)-H(x,u_2)| \leqq 4\Big(1+K_2(N)|x|\Big)|a_2|(1+|a_2|)K|x|^{N+1} \tag{2.12}$$

where $K_2(N)$ is a constant which is dependent on N. Furthermore,

$$|U[\phi](x)| \leqq \left|\phi\Big(X(x,g_N(x)+\phi(x))\Big)\right| \int_0^1 \left\{1+|x|\Big(|r_{11}|+K_3(N)|x|\Big)\right\}dt$$

where $K_3(N)$ is a constant which is dependent on N. Here we take δ sufficiently small such that $K_3(N)|x| < 1$, for $x \in D(\kappa,\delta)$, we have the (2.7) in before. Put $\theta = \arg[x]$, then $|\theta| < \kappa < \pi/2$, and $|x|\cos\theta > |x|\cos\kappa$. Since $a_2+c_{20} < 0$, if δ is sufficiently small, then

$$|v| \leqq |x|\cdot(1-\frac{1}{2}|a_2+c_{20}|\cdot|x|\cos\kappa) \leqq |x|. \tag{2.13}$$

Hence

$$\left|\phi\Big(X(x,g_N(x)+\phi(x))\Big)\right| \leqq K|x|^N\Big(1-\frac{N}{3}|a_2+c_{20}|\cdot|x|\cos\kappa\Big),$$

for sufficiently small δ. Thus,

$$|U[\phi](x)| \leqq K|x|^N\Big(1-\frac{N}{3}|a_2+c_{20}|\cdot|x|\cos\kappa\Big)\Big(1+(|r_{11}|+1)|x|\Big). \tag{2.14}$$

From (2.11), (2.12) and (2.14), we have

$$\begin{aligned} |T[\phi](x)| \leqq K|x|^N\Bigg\{ \Bigg(&\frac{K_1(N)}{K} + 4\Big(1+K_2(N)\delta\Big)|a_2|(1+|a_2|) + (|r_{11}|+1) \\ &- \frac{N}{3}|a_2+c_{20}|\cos\kappa\Big(1+(|r_{11}|+1)|x|\Big)\Bigg)|x| + 1\Bigg\}. \end{aligned}$$

If we take N to be large enough, then $\frac{N}{3}|a_2 + c_{20}| \cos\kappa\Big(1 + (|r_{11}|+1)|x|\Big) > A > 0$, for a positive constant A. Thus

$$|T[\phi](x)| \leqq K|x|^N \left\{ \left(\frac{K_1(N)}{K} + 4\Big(1 + K_2(N)\delta\Big)|a_2|(1+|a_2|) + (|r_{11}|+1) - A \right)|x| + 1 \right\}$$

Let A be sufficiently large, i.e., N be large , then we take δ small enough such that

$$K_2(N)\delta < \frac{A + (|r_{11}|+1)}{4|a_2|(1+|a_2|)} - 1, \tag{2.15}$$

i.e., $A - 4|a_2|(1+|a_2|)(1+K_2(N)\delta) + (|r_{11}|+1) > 0$, for the constant $K_2(N)$.

For the N and δ which satisfy the condition (2.15), we take K sufficiently large such that

$$K > \frac{K_1(N)}{A - 4|a_2|(1+|a_2|)(1+K_2(N)\delta) + (|r_{11}|+1)},$$

then we have $|T[\phi](x)| \leqq K|x|^N$, i.e., T in (2.9) maps $\mathfrak{F}$ into $\mathfrak{F}$.

$\mathfrak{F}$ is clearly convex, and a normal family by the theorem of Montel. Since T is obviously continuous, we obtain a fixed point $\phi_N(x)$ by Schauder's fixed point theorem [4], we conclude the existence of some fixed point $\phi(x) \in \mathfrak{F}$.

2.2.3 Uniqueness of the fixed point

Next, we show the uniqueness of the fixed point ϕ. Suppose there were two fixed points $\phi_j(x) \in \mathfrak{F}$, $j = 1, 2$. then we have

$$g_N\Big(X(x, g_N(x) + \phi_j(x))\Big) + \phi_j\Big(X(x, g_N(x) + \phi_j(x))\Big) = Y(x, g_N(x) + \phi_j(x)),\ (j = 1, 2).$$

Put $v_j = v_j(x) = X(x, g_N(x) + \phi_j(x))$, $j = 1, 2$. Then

$$\begin{cases} g_N(v_1) + \phi_1(v_1) = Y(x, g_N(x) + \phi_1(x)), \\ g_N(v_1) + \phi_2(v_1) = Y(x, g_N(x) + \phi_2(x)). \end{cases} \tag{2.16}$$

$$\begin{aligned} v_1 - v_2 &= (1 + \text{higher order terms of } x)(\phi_1(x) - \phi_2(x)), \\ g_N(v_1) - g_N(v_2) &= (2a_2 x + \text{higher order terms of } x)(\phi_1(x) - \phi_2(x)), \end{aligned} \tag{2.17}$$

and

$$\phi_2(v_1) - \phi_2(v_2) = (\phi_1(x) - \phi_2(x))(1 + \text{higher order terms of } x) \int_0^1 \phi_2'(v_2 + t(v_1 - v_2))dt.$$

Put $D_1 = \overline{D(\kappa/2, (1/2)\delta)}$ and $C = \{\xi \mid |\xi - x| = r = |x|\sin\frac{\kappa}{2}, \text{ for } x \in D_1\}$. Then $C \subset D$ and by the Cauchy's integral formula, we see that, for $x \in D_1 \backslash \{0\}$,

$$|\phi_2'(x)| \leqq \frac{1}{2\pi}\int_C \frac{|\phi_2(\xi)|}{|\xi - x|^2}|d\xi| \leqq \frac{1}{2\pi}\int_C \frac{K|\xi|^N}{(|x|\sin\frac{\kappa}{2})^2}|d\xi|.$$

Since $|\xi| \leqq |x| + |\xi - x| \leqq |x|(1 + \sin\frac{\kappa}{2})$, $|\phi_2'(x)| \leqq K\frac{(1+\sin\frac{\kappa}{2})^N}{\sin\frac{\kappa}{2}}|x|^{N-1}$. Thus,

$$|\phi_2(v_1) - \phi_2(v_2)| \leqq K\frac{(1+\sin\frac{\kappa}{2})^N}{\sin\frac{\kappa}{2}}|1 + \text{higher order terms of } x| \cdot |\phi_1(x) - \phi_2(x)| \cdot |x|^{N-1}.$$

Hence, for a fixed $N > 3$,

$$|\phi_2(v_1) - \phi_2(v_2)| \leqq K_4(N)|x|^2|\phi_1(x) - \phi_2(x)|, \tag{2.18}$$

where $K_4(N)$ is a constant which is dependent on N. On the other hand,

$$\begin{aligned} Y(x, g_N(x) + \phi_1(x)) - &Y(x, g_N(x) + \phi_2(x)) \\ &= (1 + d_{11}x + \text{higher order terms of } x)(\phi_1(x) - \phi_2(x)). \end{aligned} \tag{2.19}$$

For $x \in D_1$, by substituting (2.17)-(2.19) into (2.16), we have

$$\phi_1(v_1) - \phi_2(v_1) = (1 + (d_{11} - 2a_2)x - K_4(N)x^2 + O(x^2))(\phi_1(x) - \phi_2(x)).$$

Write $h(x) = 1 + (d_{11} - 2a_2)x - K_4(N)x^2 + O(x^2)$, then

$$\phi_1(v_1) - \phi_2(v_1) = h(x)(\phi_1(x) - \phi_2(x)). \tag{2.20}$$

Next, for sufficiently small δ, we have $\frac{|x|}{2} < |x|(1 - |a_2 + c_{20}||x|(1 + \frac{\cos\kappa}{2}))$. Since $\cos\kappa < 1 + \frac{\cos\kappa}{2}$, further from (2.13), if we let $p_1 = |a_2 + c_{20}|(1 + \frac{1}{2}\cos\kappa) > 0$ and $p_2 = \frac{1}{2}|a_2 + p_{20}|\cos\kappa$, we have

$$|x|(1 - p_1|x|) \leqq |v_1(x)| \leqq |x|(1 - p_2|x|), \tag{2.21}$$

for sufficiently small x. In the case where $x \in D(\kappa, \delta)$, then $v_1 \in D(\kappa, \delta)$, and hence, the following estimations hold:

$$|v_1^{n-1}(x)|(1 - p_1|v_1^{n-1}(x)|) \leqq |v_1^n(x)| \leqq |v_1^{n-1}(x)|(1 - p_2|v_1^{n-1}(x)|), \ (n \geqq 1) \tag{2.22}$$

where $v_1^{k+1}(x) = v(v^k(x))$, $v_1^0(x) = x$. From these inequalities, we have

$$|x|\prod_{k=0}^{n-1}(1 - p_1|v_1^k(x)|) \leqq |v_1^n(x)| \leqq |x|(1 - p_2|x|)\prod_{k=1}^{n-1}(1 - p_2|v_1^k(x)|). \tag{2.23}$$

On the other hand, from the condition (1.4), we have $d_{11} - 2a_2$, hence, if we take δ sufficiently small, then we have $|h(x)| \geqq 1 - 2|d_{11} - 2a_2| \cdot |x|$. Put $b = 2|d_{11} - 2a_2| > 0$, from (2.20), we have the following inequalities:

$$|\phi_1(v_1^n(x)) - \phi_2(v_1^n(x))| \geqq (1 - b|v_1^{n-1}(x)|) \cdot |\phi_1(v_1^{n-1}(x)) - \phi_2(v_1^{n-1}(x))|, (n \geqq 1).$$

From these, we have

$$|\phi_1(x) - \phi_2(x)| \leqq \frac{|\phi_1(v_1^n(x)) - \phi_2(v_1^n(x))|}{\prod_{k=0}^{n-1}(1 - b|v_1^k(x)|)}. \tag{2.24}$$

From the definition of ϕ_1 and ϕ_2, we have

$$|\phi_1(v_1^n(x)) - \phi_2(v_1^n(x))| \leqq 2K|v_1^n(x)|^N, \quad (n = 0, 1, 2, \cdots, n-1).$$

Similarly, from (2.23) and (2.24), we have

$$|\phi_1(x) - \phi_2(x)| \leqq 2K|x|^N \prod_{k=0}^{n-1} \frac{(1 - p_2|v_1^k(x)|)^N}{1 - b|v_1^k(x)|}.$$

Furthermore, we can take N sufficiently large, for a given δ, such that $p_2 N - p_1 - b \geqq 0$. Then we have

$$(1 - p_1|v_1^k(x)|) - \frac{(1 - p_2|v_1^k(x)|)^N}{1 - b|v_1^k(x)|} \geqq 0.$$

Here, we put $q(t) = t(1 - p_1 t)$, $r_0 = r = |x|$, $r_k = q^k(t) = q(q^{k-1}(t)) = q(r_{k-1})$, $k \geqq 2$ and $r_1 = q(t)$. From (2.21) and (2.22), by induction, we have $|v_1^k(t)| \leqq r_{k-1}$, $(r_0 = r = |x|)$. Note that $q'(t) = 1 - 2p_1 t$, $q''(t) = -2p_1$, thus for $0 \leqq t < \frac{p_1}{2}$, we have $0 < q'(t) < 1$ and $q''(t) < 0$. Then, making use of [1], for $r < \frac{1}{2p_1}$, $r_n = q^n(r) \to 0$, (as $n \to \infty$). Hence, from (2.23), we have

$$|x| \prod_{k=0}^{n-1} (1 - p_1|v_1^k(x)|) \leqq |v_1^n(x)| \leqq r_{n-1} \to 0, (\text{ as } n \to 0).$$

Thus,

$$|\phi_1(x) - \phi_2(x)| \leqq 2K|x|^{N-1}|x| \prod_{k=0}^{\infty} (1 - p_1|v_1^k(x)|) = 0.$$

Therefore,

$$\phi_1(x) \equiv \phi_2(x) \text{ for } x \in D(\kappa, \delta).$$

From the above discussion, if N is fixed, then there can only be a unique solution $\phi_N(x)$ which is dependent on N such that

$$\Psi_N(x) - g_N(x) = \phi_N(x), \quad |\phi_N(x)| \leqq K_N|x|^N,$$

where Ψ_N is a solution of (1.1).

2.2.4 Independence of N

Let $\Psi_{N'}$ and Ψ_N, $(N' > N)$ be solutions of (1.1). Put $\delta = \min(\delta_N, \delta'_N)$ and

$$\Psi_{N'}(x) = g_{N'}(x) + \phi_{N'}(x) = g_N(x) + \Big(g_{N'}(x) - g_N(x) + \phi_{N'}(x)\Big), \text{ for } x \in D(\kappa, \delta).$$

From the uniqueness of $\phi_{N'}$, we see that $g_{N'}(x) - g_N(x) + \phi_{N'}(x) = \phi_N(x)$, for $x \in D(\kappa, \delta)$. Then we can define $\Psi_{N,N'}$ as

$$\Psi_{N,N'} = \begin{cases} \Psi_N & \Big(x \in D(\kappa, \delta_N)\Big), \\ \Psi_{N'} & \Big(x \in D(\kappa, \delta'_N)\Big), \end{cases}$$

and if $\delta = \min(\delta_N, \delta_{N'})$, we see that

$$\Psi_{N'} = \Psi_N \text{ for } x \in D(\kappa, \delta).$$

In that way, we can obtain a solution Ψ of (1.1), which is independent of N.

2.2.5 Solutions of the equation (1.1)

Take $N' = N+1$ and $\delta = \min(\delta_N, \delta_{N'})$ in the subsection 2.2.4. Then, for $x \in D(\kappa, \delta)$,

$$|\phi_{N'}(x)| = |\Psi_{N+1}(x) - g_{N+1}(x)| = |\Psi_N(x) - g_{N+1}(x)| \leqq (K_N + |a_{N+1}|)|x|^{N+1}.$$

We put $C_N = K_N + |a_{N+1}|$. Then we have

$$|\Psi(x) - g_N(x)| \leqq C_N |x|^{N+1}, \text{ for } x \in D(\kappa, \delta),$$

where C_N is a constant and δ is sufficiently small.

This also completes the proof of Theorem 1.

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